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Solution of a one-dimensional stochastic model with branching and coagulation reactions

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We solve a one-dimensional stochastic model of interacting particles on a chain. Particles can have branching and coagulation reactions; they can also appear on an empty site and disappear spontaneously. This model, which can be viewed as an epidemic model and/or as a generalization of the *voter* model, is treated analytically beyond the *conventional* solvable situations. With help of a suitably chosen *string function*, which is simply related to the density and the noninstantaneous two-point correlation functions of the particles, exact expressions of the density and of the noninstantaneous two-point correlation functions, as well as the relaxation spectrum are obtained on a finite and periodic lattice.

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Due to their important role in the description of classical interacting many-particle nonequilibrium systems, reaction-diffusion (RD) models have been extensively investigated in the last decade [1,2]. In lower dimensions, they provide relevant examples of *strongly correlated* systems which cannot be correctly described by mean-field-like approaches. In this sense satisfying comprehension of RD models in lower dimensions would require *exact solutions*, which are scarce, even in one spatial dimension. In some cases, however, certain RD models are known to be solvable. These cases can essentially be classified into four categories: (i) models for which the equations of motion of correlation functions are closed [3]; (ii) the *free-fermion* models [4] (or systems which can be mapped onto the latter, see Refs. [2,5]); some other (one-dimensional) RD models can be solved by the (iii) *Matrix Ansatz* method [6], and some others by (iv) the *interparticle distribution function* (IPDF) method [7–9], first introduced for the study of the diffusion-coagulation model (and its variants). It has also to be mentioned that the solutions of various one-dimensional RD models have been obtained from the diffusion-coagulation models via *similarity transformations* [2,10]. It has been established that the latter solvable situations correspond to *free-fermion* systems [2].

The purpose of this Rapid Communication is to present a generalization of the IPDF method and to apply this technique to solve a one-dimensional stochastic model which is not solvable using *conventional* methods. The model under consideration exhibits a *massive* spectrum, implying an *exponential approach* towards the steady state. The expressions of the density and noninstantaneous correlation functions are determined.

Consider a periodic lattice of L sites on which (classical) particles interact. Each site is either empty (denoted by the symbol $\emptyset$) or occupied by a particle at most, say, of species $\mathbb{A}$ (*hard-core interaction*). When a particle and a vacancy are adjacent to each other, a *branching reaction* can take place and the particle $\mathbb{A}$ can give birth to an offspring ($\mathbb{A}\emptyset\rightarrow\mathbb{A}\mathbb{A}$ and $\emptyset\mathbb{A}\rightarrow\mathbb{A}\mathbb{A}$) with rate $\Gamma_{10}^{11}=\Gamma_{01}^{11}$; another possible reaction is the *death* of the particle ($\mathbb{A}\emptyset\rightarrow\emptyset\emptyset$ and $\emptyset\mathbb{A}\rightarrow\emptyset\emptyset$) with rate $\Gamma_{10}^{00}=\Gamma_{01}^{00}$. When two particles are adjacent, they can *coagulate* ($\mathbb{A}\mathbb{A}\rightarrow\mathbb{A}\emptyset$ and $\mathbb{A}\mathbb{A}\rightarrow\emptyset\mathbb{A}$) with rate $\Gamma_{11}^{10}=\Gamma_{11}^{01}$. In addition, when two vacancies are adjacent, a particle can appear (*birth* process, $\emptyset\emptyset\rightarrow\mathbb{A}\emptyset$ and $\emptyset\emptyset\rightarrow\emptyset\mathbb{A}$) with rate $\Gamma_{00}^{10}=\Gamma_{00}^{01}$. The system described above can be viewed as an *epidemic model*, where particles can spontaneously appear or disappear, have an offspring, and coagulate. It can also be viewed as a generalization of the *voter* model [2], where the presence or absence of a particle is associated with an opinion (yes or no) and each site is associated with a human being. According to the dynamics of the model, each individual changes his opinion at a rate proportional to the opinion of his neighbors.

A particle (vacancy) at each of the L-lattice sites corresponding to spin down (up), the master equation of the model, can be rewritten as an imaginary-time Schrödinger equation for a *quantum spin-chain* problem: $\partial/\partial t|P(t)\rangle=-H|P(t)\rangle$, where $|P(t)\rangle=\Sigma_{\{n\}}P(\{n\},t)|\{n\}\rangle$ describes the state of the system at time t (the sum runs over the 2^L configurations) and H is the *stochastic Hamiltonian* (non-Hermitian) expressed in a spin-$\frac{1}{2}$ representation as $H=\Sigma_{j=1}^{L}H_{j,j+1}$, with

$$\begin{aligned}-H_{j,j+1}=&\Gamma_{10}^{00}\{(1-n_{j+1})(\sigma_j^{+}-n_j)\\&+(1-n_j)(\sigma_{j+1}^{+}-n_{j+1})\}+\Gamma_{00}^{10}\{(1-n_{j+1})\\&\times(\sigma_j^{-}+n_j-1)+(1-n_j)(\sigma_{j+1}^{-}+n_{j+1}-1)\}\\&+\Gamma_{11}^{10}\{n_j(\sigma_{j+1}^{+}-n_{j+1})+n_{j+1}(\sigma_j^{+}-n_j)\}\\&+\Gamma_{10}^{11}\{n_{j+1}(\sigma_j^{-}+n_j-1)+n_j(\sigma_{j+1}^{-}+n_{j+1}-1)\},\end{aligned}\tag{1}$$

where the $\sigma^{\pm}$ are the usual Pauli matrices and $n_j\equiv\frac{1}{2}(1-\sigma_j^z)$ is the density operator at site j. We also define the "left vacuum" $\langle\tilde{\chi}|\equiv\Sigma_{\{n\}}\langle\{n\}|$. The probability conservation yields $\langle\tilde{\chi}|H=0$.

For the model under consideration, the equation of evolution of the density, from an initial state $|P(0)\rangle$, is therefore

$$\begin{aligned}\frac{d}{dt}\langle n_j\rangle=&-\langle\tilde{\chi}|n_j(H_{j-1,j}+H_{j,j+1})e^{-Ht}|P(0)\rangle\\=&2A+B(\langle n_{j+1}\rangle+\langle n_{j-1}\rangle)-2C\langle n_j\rangle\\&+D(\langle n_jn_{j+1}\rangle+\langle n_jn_{j-1}\rangle),\end{aligned}\tag{2}$$

where $A\equiv\Gamma_{00}^{10}$, $B\equiv\Gamma_{10}^{11}-\Gamma_{00}^{10}$, $C\equiv\Gamma_{10}^{00}+\Gamma_{00}^{10}$, and $D\equiv\Gamma_{10}^{00}+\Gamma_{00}^{10}-(\Gamma_{10}^{11}+\Gamma_{11}^{10})$. When $D=0$ and $B\neq C$, for a translationally invariant system with initial density of particles $\langle n_j(0)\rangle=\rho(0)$, the solution of Eq. (2) simply reads $\langle n_j(t)\rangle=A/\{C-B\}+[\rho(0)-(A/\{C-B\})]e^{-2(C-B)t}$. However, when $D\neq 0$, it is clear from Eq. (2) that the equation of motion of the correlation functions of the model give rise to an open hierarchy [3,2], which is not, in general, solvable. In addition, Hamiltonian (1) is not quadratic and cannot, in general (excepted when $\Gamma_{10}^{11}=\Gamma_{10}^{00}$ and $\Gamma_{11}^{10}=\Gamma_{00}^{10}$, see Ref. [2] for a complete classification of *free-fermion* systems), be casted into a free-fermion form. Furthermore, this model cannot be solved by the *traditional* IPDF method, which is not applicable [7–9] in the presence of the processes $\mathbb{A}\emptyset\to\emptyset\emptyset$; $\emptyset\mathbb{A}\to\emptyset\emptyset$ and in the absence of processes $\mathbb{A}\emptyset\to\emptyset\mathbb{A}$; $\emptyset\mathbb{A}\to\mathbb{A}\emptyset$ (the latter should occur with the same rate as the coagulation rates [7–9]).

To our knowledge, no *exact results* have been obtained for the model under investigation beyond the $D=0$ and *free-fermion* cases. In order to obtain the exact expression of the density beyond the latter *conventional cases*, we generalize the IPDF method introducing the following *string function* $(L\geqslant y\geqslant x\geqslant 1)$:

$$S_{x,y}(t)\equiv\langle(a-bn_x)(a-bn_{x+1})\dots(a-bn_{y-1})\rangle(t), \tag{3}$$

where a and b are nonvanishing numbers. When $a=b=1$, $S_{x,y}(t)$ is the *empty interval function* used in the *traditional* IPDF method [7–9]. The idea to solve the model under consideration here (with certain restrictions for the reaction rates) is to *choose* suitable a and b in order to have a closed equation of evolution for $S_{x,y}(t)$. This is achieved by imposing the following ratio between a and b:

$$\frac{b}{a}=1+\frac{\Gamma_{11}^{10}}{\Gamma_{00}^{10}}>1 \tag{4}$$

and for the following reaction rates:

$$\Gamma_{11}^{10}=\Gamma_{11}^{01}>0;\quad 2\Gamma_{00}^{10}=2\Gamma_{00}^{01}\geqslant\Gamma_{10}^{11}=\Gamma_{01}^{11}\geqslant 0;$$

$$\text{and}\quad \Gamma_{10}^{00}=\Gamma_{01}^{00}=\frac{\Gamma_{11}^{10}(2\Gamma_{00}^{10}-\Gamma_{10}^{11})}{\Gamma_{00}^{10}}\geqslant 0. \tag{5}$$

According to Eq. (4) and with rates (5), for the model under consideration on a periodic lattice of L sites, we have $(1\leqslant x\leqslant y\leqslant L)$:

$$\frac{d}{dt}S_{x,y}(t)=\frac{\alpha}{2}(S_{x+1,y}(t)+S_{x,y-1}(t))+\frac{\beta}{2}(S_{x-1,y}(t)+S_{x,y+1}(t))-\gamma S_{x,y}(t)-(y-x)\,\delta S_{x,y}(t)\quad(1\leqslant x<y<L),$$

$$\frac{d}{dt}S_{x,x+L}(t)=-L\,\delta S_{x,x+L}(t), \tag{6}$$

$$S_{x,x}(t)=1,$$

where $\alpha\equiv 2(aC-bA)$, $\beta\equiv-2D/b$, $\gamma\equiv 2(B+C)-\delta$, and $\delta\equiv(2b/a)A>0$. The prescription $S_{x,x}(t)=1$ is obtained requiring that $S_{x,x+1}(t)=a-b\langle n_x(t)\rangle$ and using Eq. (2).

The subcase $\Gamma_{10}^{11}=\Gamma_{00}^{10}$ implies $\alpha=\beta=B=D=0$ and we recover $(C\neq 0)$ $\langle n_x(t)\rangle=[a-S_{x,x+1}(t)]/b=A/C+(\langle n_x(0)\rangle-A/C)e^{-2Ct}$.

Hereafter we focus on the more general situation where Eqs. (5) are fulfilled with $\Gamma_{10}^{11}\neq\Gamma_{00}^{10}$, and thus $\alpha\neq 0,\beta\neq 0$.

It is useful to consider the *auxiliary* function $\mathcal{R}_{x,y}(t)\equiv\mu^{x-y}S_{x,y}(t)$, where we introduce the complex numbers $\mu\equiv-i\,\mathrm{sgn}(\alpha)|\alpha/\beta|^{1/2}$ and $q\equiv i|\alpha\beta|^{1/2}\neq 0$. Notice that, because of Eqs. (5), $0<|q|/\delta<1/2$. With help of Eqs. (5), we obtain the equation of motion of $\mathcal{R}_{x,y}(t)$,

$$\frac{d}{dt}\mathcal{R}_{x,y}(t)=\frac{q}{2}\sum_{e=\pm 1}\{\mathcal{R}_{x+e,y}(t)+\mathcal{R}_{x,y+e}(t)\}-\gamma\mathcal{R}_{x,y}(t)-(y-x)\,\delta\mathcal{R}_{x,y}(t);\quad(1\leqslant x<y<L),$$

$$\frac{d}{dt}\mathcal{R}_{x,x+L}(t)=-L\,\delta\mathcal{R}_{x,x+L}(t), \tag{7}$$

$$\mathcal{R}_{x,x}(t)=1.$$

The stationary solution of Eq. (7) is obtained with the Ansatz $\mathcal{R}_{x,y}(\infty)=\widetilde{A}_L J_{y-x+\omega}(2q/\delta)+\widetilde{B}_L Y_{y-x+\omega}(2q/\delta)$, where $J_\nu(z)$ and $Y_\nu(z)$ are the usual Bessel functions of first and second kind, respectively, and $\widetilde{A}_L$ and $\widetilde{B}_L$ are constants to be determined. Inserting the expression of $\mathcal{R}_{x,y}(\infty)$ into Eq. (7), we obtain $\omega=\gamma/\delta$. Taking into account the boundary conditions $\mathcal{R}_{x,x}(t)=\mathcal{R}_{x,x}(\infty)=1$ and $\mathcal{R}_{x,x+L}(\infty)=0$, we get

$$\widetilde{A}_L=-\frac{Y_{L+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}, \tag{8}$$

$$\widetilde{B}_L=\frac{J_{L+\gamma/\delta}(2q/\delta)}{J_{L+\gamma/\delta}(2q/\delta)Y_{\gamma/\delta}(2q/\delta)-Y_{L+\gamma/\delta}(2q/\delta)J_{\gamma/\delta}(2q/\delta)}, \tag{9}$$

which provides the stationary expression for the string function:

$$S_{x,y}(\infty)=\mu^{y-x}[\widetilde{A}_L J_{y-x-\gamma/\delta}(2q/\delta)+\widetilde{B}_L Y_{y-x-\gamma/\delta}(2q/\delta)]. \tag{10}$$

According to the definition of the string function, the density of particles at site x is given by $\langle n_x(t)\rangle=[a-S_{x,x+1}(t)]/b$ and therefore the explicit stationary density of particles reads

$$\langle n_x(\infty)\rangle=\frac{a-S_{x,x+1}(\infty)}{b}$$
$$=\frac{1}{b}\{a-\mu[\tilde{A}_L J_{1+\gamma/\delta}(2q/\delta)+\tilde{B}_L Y_{1+\gamma/\delta}(2q/\delta)]\}. \quad (11)$$

In order to solve the dynamical part of Eq. (7), we seek a solution of the form $\mathcal{R}_{x,y}(t)-\mathcal{R}_{x,y}(\infty)=\Sigma_\lambda r^\lambda_{y,x}e^{-\lambda qt}$. Thus Eq. (7), for $1\leqslant x<y<L$ gives rise to the following difference equation: $r^\lambda_{y-1,x}+r^\lambda_{y+1,x}+r^\lambda_{y,x-1}+r^\lambda_{y,x+1}=2\{[\gamma+(y-x)\delta]/q-\lambda\}r^\lambda_{y,x}$. With the notation $E\equiv(q\lambda-\gamma)/\delta$, this equation admits $r^\lambda_{y,x}=\mathcal{A}J_{y-x-E}(2q/\delta)+\mathcal{B}Y_{y-x-E}(2q/\delta)$ as a solution, where $\mathcal{A}$, $\mathcal{B}$, and the spectrum $\{E_i\}$ are determined from the boundary and the initial conditions. Indeed, the boundary conditions $\mathcal{R}_{x,x}(t)=1$ and $(d/dt)\mathcal{R}_{x,x+L}(t)=-L\delta\mathcal{R}_{x,x+L}(t)$ require, respectively, $r^\lambda_{x,x}=\mathcal{A}J_{-E}(2q/\delta)+\mathcal{B}Y_{-E}(2q/\delta)=0$ and $\Sigma_\lambda(q\lambda-\delta L)e^{-\lambda qt}[\mathcal{A}J_{L-E}(2q/\delta)+\mathcal{B}Y_{L-E}(2q/\delta)]=0$, i.e.,

$$\mathcal{A}J_{-E}(2q/\delta)+\mathcal{B}Y_{-E}(2q/\delta)=0,$$
$$\mathcal{A}J_{L-E}(2q/\delta)+\mathcal{B}Y_{L-E}(2q/\delta)=0. \quad (12)$$

The only nontrivial solution of this system (for which $\mathcal{A}\neq0$ and $\mathcal{B}\neq0$) requires

$$J_{L-E}(2q/\delta)Y_{-E}(2q/\delta)-J_{-E}(2q/\delta)Y_{L-E}(2q/\delta)=0, \quad (13)$$

or equivalently in terms of *Lommel function* [12],

$$R_{L-1,1-E}(2i|q|/\delta)=0. \quad (14)$$

Thus, the relaxation spectrum of the string-function of the model is obtained as the zeroes of the *Lommel function* (14). The latter admits $(L-1)$ zeroes which are symmetrically distributed around $L/2$ (which is also an eigenvalue if L is even) and have a degeneracy L. To obtain the complete set of $L(L-1)+1$ *eigenvalues*, i.e., the relaxation spectrum $\{E_i\}$, $i=1,\ldots,L$ of the *string-function* [and not the spectrum of the *Hamiltonian* (1)], we have also to take into account the *eigenvalue* $q\lambda=L\delta$, which follows directly from the boundary condition $(d/dt)\mathcal{R}_{x,x+L}(t)=-L\delta\mathcal{R}_{x,x+L}(t)$.

To our knowledge there are no explicit results on the zeroes of the Lommel function of imaginary arguments. In order to have more explicit information on the spectrum, we use the formal analogy, first noticed by Peschel *et al.* [9], which exists between the problem under consideration and the energy spectrum of an electron in a finite one-dimensional crystal in an electric potential of strength $\mathcal{E}n$ (here $\mathcal{E}=1$) [11].

To compute explicitly $\{E_i\}$, $i=1,\ldots,L$, we take advantage of the following eigenvalue-problem:

$$(E-n)F_n=V(F_{n-1}+F_{n+1});\ \ (1\leqslant n<L),$$
$$F_0=F_L=0, \quad (15)$$

where $F_n=V^E[J_{E-n}(2V)J_{-E}(2V)-(-1)^nJ_E(2V)J_{n-E}(2V)]$ are eigenfunctions. The eigenvalues of Eq. (15) are obtained as the zeroes of the following *Lommel function*: $R_{L-1,1-E}(2V)=0$ [11]. Choosing $V=i|q|/\delta$, the problem of determining the relaxation spectrum is reformulated as that of solving eigenvalue-problem (15). The latter can be recasted into the following form: $\mathcal{M}|\mathcal{F}\rangle\rangle=E|\mathcal{F}\rangle\rangle$, where $\mathcal{M}$ is a $(L-1)\times(L-1)$ symmetric (in fact *anti-Hermitian*) tridiagonal matrix and $|\mathcal{F}\rangle\rangle$ is a $(L-1)$-component column-vector: $|\mathcal{F}\rangle\rangle\equiv(F_{n=1}\ F_2\cdots F_{L-1})^T$. The general form of the matrix $\mathcal{M}$ is the following:

$$\mathcal{M}=\begin{pmatrix} 1 & V & 0 & \ldots & \ldots & \ldots & 0\\ V & 2 & V & 0 & \ldots & \ldots & 0\\ 0 & V & 3 & V & 0 & \ldots & 0\\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \ldots & 0 & \ddots & V & (L-2) & V\\ 0 & \ldots & \ldots & \ldots & 0 & V & (L-1)\end{pmatrix}. \quad (16)$$

For *small* systems the $(L-1)$ distinct eigenvalues $\{E_i\}$ of Eq. 16 can be computed analytically. For $L=6$, we have $\{E_i\}=\{3,3\pm\sqrt{(5+4V^2\pm\sqrt{9+24V^2+4V^4})/2}\}$, where we still have to take into account the additional eigenvalue $q\lambda=L\delta$. For larger matrices we had to proceed numerically. Our analysis (based on the spectrum of large matrices, with $L\leqslant1000$), shows that the spectrum $\{E_i\}$ (and therefore $\{q\lambda\}$) is *real* and symmetric around $L/2$ which is an eigenvalue when L is even. The other eigenvalues are not generally integers, but for the *central* part of the spectrum (when eigenvalues which are close of $L/2$), the eigenvalues approach integer values. This is not the case at the extremities of the spectrum. In particular, the smallest eigenvalue $E^*=\min_{E_i}\{E_i\}$ is not an integer and depends on the size of the system: $E^*=\epsilon_L>1$. However, for $L\gg1$, $\epsilon_L\rightarrow\epsilon_\infty$, and E^* is a constant: $E^*=\epsilon_\infty>1$. For $L=6$, we have the exact result $\epsilon_{L=6}=3-\sqrt{(5+4V^2+\sqrt{9+24V^2+4V^4})/2}$, with $1<\epsilon_{L=6}<3-\frac{1}{2}\sqrt{8+\sqrt{13}}$. This expression can be considered as an excellent approximation to systems of size $L\gg1$, and in particular for ϵ_∞. As an illustration, for the case $\Gamma^{10}_{00}=3/10$, $\Gamma^{11}_{10}=1/2$, $\Gamma^{10}_{11}=1$, and $\Gamma^{00}_{10}=1/3$, with the expression above, we obtain (analytically) $\epsilon_{L=6}=1.082\,333\,768\,3$. For larger systems ($L=10,25,40,1000$), we obtain numerically (with an accuracy of 10^{-10}): $\epsilon_{10}=\epsilon_{25}=\epsilon_{40}=\epsilon_{1000}=1.082\,333\,769\,7$.

Therefore, the long-time dynamics (of large systems, with $L\gg1$) is governed by the eigenvalue $E^*=\epsilon_L\approx3-\sqrt{(5+4V^2+\sqrt{9+24V^2+4V^4})/2}$, i.e.,

$$q\lambda^*=E^*\delta+\gamma=\epsilon_L\delta+\gamma$$
$$=2\left[\frac{\Gamma^{10}_{00}\Gamma^{11}_{10}+\Gamma^{10}_{11}(2\Gamma^{10}_{00}-\Gamma^{11}_{10})}{\Gamma^{10}_{00}}+(\epsilon_L-1)(\Gamma^{10}_{00}+\Gamma^{10}_{11})\right]$$
$$>2\Gamma^{11}_{10}$$
$$\geqslant0. \quad (17)$$

Equation (17) provides the *inverse of the relaxation-time* of the system [13].

With the knowledge of the spectrum $\{E_i\}, i=1,\ldots,L$, the expression of the density at site x reads

$$\langle n_x(t)\rangle-\langle n_x(\infty)\rangle =\frac{\mu}{b}\sum_{E_i}\mathcal{A}_{E_i}e^{-(E_i\delta+\gamma)t}[Y_{1-E_i}(2q/\delta)J_{L-E_i}(2q/\delta) -J_{1-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta)]. \tag{18}$$

The coefficients $\mathcal{A}_{E_i}$ are obtained from the initial condition [in the translationally-invariant situation, where $S_{x,y}(t)=S_{y-x}(t)$] according to

$$\mathcal{A}_{E_i}=\sum_{j,n=1}^{L}[\mathcal{N}^{-1}]_{i,j}[J_{n-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) -Y_{n-E_j}(2q/\delta)J_{L-E_j}(2q/\delta)]^*[S_n(0)-S_n(\infty)]\mu^{-n}, \tag{19}$$

where $\mathcal{N}$ is a Hermitic $L\times L$ matrix whose entries read (for details, see Ref. [13]):

$$\mathcal{N}_{i,j}=\sum_{n=1}^{L}(J_{n-E_i}(2q/\delta)Y_{L-E_i}(2q/\delta)-Y_{n-E_i}(2q/\delta) \times J_{L-E_i}(2q/\delta))^*(J_{n-E_j}(2q/\delta)Y_{L-E_j}(2q/\delta) -Y_{n-E_j}(2q/\delta)J_{L-E_j}(2q/\delta)).$$

The long-time behavior of $\langle n_x(t)\rangle$ is obtained in retaining in Eq. (18) only the term $E_i=E^*$.

Result (18) can be extended to the computation of $\langle n_x(t)n_{x_0}(0)\rangle$. To do this it suffices to take $n_{x_0}|P(0)\rangle$ (instead of $|P(0)\rangle$) as the initial state in Eq. (18) and thus to replace the coefficients $\mathcal{A}_{E_i}$ of Eq. (19) by those computed in considering $\langle\{\prod_{j=x}^{y-1}[a-bn_j(0)]\}n_{x_0}(0)\rangle$ instead of $S_{y-x}(0)$.

In this work we have proposed a natural generalization of the IPDF method to solve (with some restrictions on the reaction-rates) a one-dimensional reaction-diffusion model which can be viewed as an epidemic model and/or a generalization of the voter model and that could not be solved by previous approaches.

On a finite and periodic lattice, we have obtained the exact expression of the steady state, of the dynamical part of the density, and of the noninstantaneous two-point correlation functions of the model under consideration, which exhibits a *massive* and *real* relaxation spectrum. This means that steady-state (11) of the system is reached exponentially with a relaxation-time which is determined explicitly in Eq. (17).

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